On CAP Representations of Even Orthogonal Groups

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Outline

Ongoing joint work with David Ginzburg and Dihua Jiang.

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SO_N denotes the split special split orthogonal group in N variables, viewed as an algebraic group over F.

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[a] Assume that π is CAP with respect to

$$Ind_{Q_m(\mathbb{A})}^{\mathrm{SO}_{2(m+k)}(\mathbb{A})} \tau |\det \cdot|^{s_0} \otimes \sigma, \quad s_0 \geq 0,$$

where τ, σ are unitary, irreducible, automorphic, cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$, $\mathrm{SO}_{2k}(\mathbb{A})$, respectively.

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which cannot be an image of a functorial lift of a cuspidal generic representation.

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[b] Let $m = 2$. Assume that π (on $\mathrm{SO}_{2(2+k)}(\mathbb{A})$) supports Fourier coefficients corresponding to the sub-regular nilpotent orbit; it corresponds to the partition $(2k + 1, 3)$.

The corresponding unipotent group and character have the form

$$\begin{pmatrix} z & y & * & * & * \\ & I_2 & x & * & * \\ & & I_2 & x' & * \\ & & & I_2 & y' \\ & & & & z^* \end{pmatrix} \mapsto \psi_{Z_{k-1}}(z)\psi(y_{k-1,1} - \frac{\beta}{2} y_{k-1,2})\psi(tr(x));$$

$\beta \in F^*$,

$$z \in Z_{k-1} = \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & & * \\ & & \cdots & \\ & & & 1 \end{pmatrix}_{k-1 \times k-1} \qquad ;$$

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$$\psi_{Z_{k-1}}(z) = \psi(z_{1,2} + z_{2,3} + \cdots z_{k-2,k-1}).$$

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3. $L^S(\pi \times \tau, s)$ *has a pole at* $s = \frac{3}{2}$ *(maximal).*

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[b'] $\mathcal{O}(\pi) = (2k + 2n - 1, 2n + 1)$ $(m = 2n)$.
We prove that τ is symplectic (and (2), (3) above).
A Fourier coefficient with respect to $(2k + 2n - 1, 2n + 1)$ corresponds to the following unipotent subgroup and character

$$\begin{pmatrix} z & y & * & \cdots & * & * & * & \cdots & * & * \\ & I_2 & x_1 & & * & * & * & & * & * \\ & & & \cdots & & & & & & \\ & & & & I_2 & x_n & * & \cdots & * & * \\ & & & & & I_2 & x_n' & & & * \\ & & & & & & \cdots & & & \\ & & & & & & & & x_1' & * \\ & & & & & & & & I_2 & y' \\ & & & & & & & & & z^* \end{pmatrix}$$

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$$\mapsto \psi_{Z_{k-1}}(z)\psi(y_{k-1,1} - \frac{\beta}{2}y_{k-1,2})\psi(tr(x_1 + \cdots x_n))$$

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We represent $L^S(\pi \times \tau, s)$ by Rankin-Selberg integrals in terms of Gelfand-Graev coefficients of π paired against π'.
They involve an Eisenstein series on $\mathrm{SO}_{6n+1}(\mathbb{A})$, induced from $\tau|\det \cdot|^s \otimes \pi'$. In fact, the Rankin-Selberg integral, in this case, is a Gelfand-Graev coefficient of this Eisenstein series paired against π (or vice versa).

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Eventually, $L(\pi' \times \tau, s)$ has a pole at $s = \frac{1}{2} + s_0$. Hence $s_0 = \frac{1}{2}$, τ is symplectic and π' lifts to τ.

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The corresponding unipotent group and character are of the form

$$\begin{pmatrix} z & x & y \\ & I_{2n'+2} & x' \\ & & z^* \end{pmatrix}$$

$$\mapsto \psi_{Z_{m+k-n'-1}}(z)\psi(x_{m+k-n'-1,n'+1} - \frac{\alpha}{2}x_{m+k-n'-1,n'+2}),$$

$z \in Z_{m+k-n'-1}, \quad \alpha \in F^*$.
This is the information we need, so that we can use the Rankin-Selberg integrals, as above.

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Hence $s_0 = \frac{1}{2}$, τ is symplectic, and, in particular, $m = 2n$ must be even. Also, it follows that $n \leq n'$ and τ figures in the isobaric sum, which is the lift of π' to $\mathrm{GL}_{2n'}(\mathbb{A})$.

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$$\sigma_v = Ind_{B_{SO_{2k}}(F_v)}^{SO_{2k}(F_v)} \mu_1 \otimes \cdots \otimes \mu_k.$$

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Thus, π satisfying [a](with $s_0 = \frac{1}{2}$; τ symplectic, e.g. assume [b"]) is such that for any nilpotent orbit $\mathcal{O}$ supporting Fourier coefficients on π, we must have

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Hence [b'] says that π "attains" the maximum nilpotent orbit possible.

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π is not generic, and π has nontrivial Fourier coefficients with respect to the sub-regular nilpotent orbit; $\mathcal{O}(\pi) = (2k + 1, 3)$.

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What we explained before proves that, conversely, the assumption on $\mathcal{O}(\pi)$ and the fact that $L^S(\pi \times \tau, s)$ has a pole at $s = \frac{3}{2}$ imply that τ is symplectic. This is first part of the following theorem which proves the converse.

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1. τ is symplectic.
2. There is an irreducible, automorphic, cuspidal, generic representation σ of $\mathrm{SO}_{2k}(\mathbb{A})$, such that π is CAP with respect to

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Consider the Eisenstein series corresponding to

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Then $\Theta_{\tau,v}$ is the unramified constituent of

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This representation has degenerate Whittaker models with respect to a unique maximal nilpotent orbit; it corresponds to the partition $((2n)^{2k+2})$.

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$$z = \begin{pmatrix} u & y & c \\ & I_{4(n+k)} & y' \\ & & u^* \end{pmatrix},$$

where u has the form

$$u = \begin{pmatrix} I_{2k} & x_1 & \cdots & & \\ & I_{2k} & x_2 & & \\ & & \cdots & & \\ & & & & x_{n-2} \\ & & & & I_{2k} \end{pmatrix}$$

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Denote $x_{n-1} = (0_{2k\times 2k(n-2)}, I_{2k})y \begin{pmatrix} 0_{2n+k\times 2k} \\ I_{2k} \\ 0_{2n+k\times 2k} \end{pmatrix}$.

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Then $\psi_{Z_{n,k}}(z) = \psi(tr(x_1 + x_2 + \cdots + x_{n-1}))$.

The stabilizer of $\psi_{Z_{n,k}}$ in the Levi subgroup $\mathrm{GL}_{2k}^{n-1} \times \mathrm{SO}_{4(n+k)}$ is isomorphic to $\mathrm{SO}_{2k} \times \mathrm{SO}_{4n+2k}$;

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$(g, h) = diag(g, ..., g, [g, h], g, ..., g)$ (g repeated $(n-1) + (n-1)$ times);

$$[g, h] = \begin{pmatrix} h_1 & & h_2 \\ & g & \\ h_3 & & h_4 \end{pmatrix},$$

$g \in \mathrm{SO}_{2k}$, $h \in \mathrm{SO}_{4n+2k}$.

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In case $n = 1$, $f^{\psi_{Z_{n,k}}}(g,h) = f([g,h])$.

We use $f^{\psi_{Z_{n,k}}}(g, h)$ as a kernel function.

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We may start with σ on $\mathrm{SO}_{2k}(\mathbb{A})$ and define π similarly via

$$h \mapsto \int_{\mathrm{SO}_{2k}(F)\backslash \mathrm{SO}_{2k}(\mathbb{A})} f^{\psi_{Z_{n,k}}}(g,h)\varphi_\sigma(g)dg.$$

To get the correspondence of unramified parameters, we prove, at finite places v, where our representations are unramified

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Theorem

Assume that
$Hom_{\mathrm{SO}_{2k}(F_v)\times\mathrm{SO}_{4n+2k}(F_v)}(\mathcal{J}_{(\psi_v)_{Z_{n,k}}}(\Theta_{\tau,v})\otimes\hat{\sigma}_v\otimes\pi_v,1)\neq 0.$

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Then π_v *is isomorphic to the unramified constituent of*

$$Ind_{Q_{2n}(F_v)}^{\mathrm{SO}_{4n+2k}(F_v)}\tau_v|\det\cdot|^{\frac{1}{2}}\otimes\sigma_v.$$

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$$a_\ell = \begin{pmatrix} I_\ell & 0 \\ 0 & 0 \end{pmatrix}, \quad \ell \leq k.$$

Substitute the Fourier expansion. We get

$$\int_{\mathrm{SO}_{4+2k}(F)\backslash \mathrm{SO}_{4+2k}(\mathbb{A})} \varphi_\pi(h) \sum_{(\alpha,\beta)}$$

$$\int_{U_k(F)\backslash U_k(\mathbb{A})} f(\begin{pmatrix} I_k & x & y \\ & I_{4+2k} & x' \\ & & I_k \end{pmatrix} \begin{pmatrix} \alpha & & \\ & \beta h & \\ & & \alpha^* \end{pmatrix})\psi^{-1}(tr(xa_\ell))dudh,$$

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where $(\alpha, \beta) \in (\bar{P}_\ell(F) \times Q_\ell(F))^{\Delta} \backslash \mathrm{GL}_k(F) \times \mathrm{SO}_{4+2k}(F)$.

The elements of $(\bar{P}_\ell(F) \times Q_\ell(F))^\Delta$ have the form

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Collapsing sum and integral,

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Thus, when $\ell \geq 1$, we can factor our integral modulo the Adele points of the unipotent radical of Q_ℓ and the corresponding constant term of the cusp form φ_π appears as an inner integral, and hence we get zero.

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Of course, for $f' \in \Theta'_\tau$,

$$\int_{\mathrm{SO}_{4+2k}(F)\backslash \mathrm{SO}_{4+2k}(\mathbb{A})} f'(h)\varphi_\pi(h)dh = 0.$$

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where U_r is the unipotent radical of the standard parabolic subgroup of $\mathrm{SO}_{4(k+1)}$, whose Levi part is isomorphic to $\mathrm{GL}_r \times \mathrm{SO}_{4(k+1)-2r}$.

Thus, $r = 2j$ must be even and the constant term $f^{U_{2j}}(h)$ can be expressed in terms of a similar residual Eisenstein series Θ'_τ on $\mathrm{SO}_{4(k-j+1)}(\mathbb{A})$. Therefore, we need to consider, for $f' \in \Theta'_\tau$,

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This is a tower property.
We use that $\mathcal{O}(\pi) = (2k+1, 3)$ to show that the last integral is identically zero.

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$$\int_{\mathrm{SO}_8(F)\backslash \mathrm{SO}_8(\mathbb{A})} \varphi_\pi(h) \int_{F^2\backslash \mathbb{A}^2} f(\begin{pmatrix} 1 & x & & y & * \\ & 1 & & 0 & -y \\ & & h & & \\ & & & 1 & -x \\ & & & & 1 \end{pmatrix})\psi^{-1}(x+y)dxdydh$$

is not identically zero.

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We use again the fact that $((2)^{2k+2})$ is the maximal nilpotent orbit of Θ_τ to show that only the orbit of e, such that $< e, e >= -2$ contributes to the Fourier expansion.

Substitute the Fourier expansion. We get

$$\int_{\mathrm{SO}_8(F)\backslash \mathrm{SO}_8(\mathbb{A})} \varphi_\pi(h) \sum_\gamma$$

$$\int_{F^{10}\backslash \mathbb{A}^{10}} f(\begin{pmatrix} 1 & u & * \\ & I_{10} & u' \\ & & 1 \end{pmatrix} \begin{pmatrix} I_2 & & \\ & \gamma h & \\ & & I_2 \end{pmatrix})\psi^{-1}(u_1+u_2-u_9+u_{10})dudh,$$

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where $\gamma \in \mathrm{SO}_7(F)\backslash \mathrm{SO}_8(F)$ (with an appropriate embedding of SO_7 inside SO_8) .

We can rewrite the last integral as

$$\int_{\mathrm{SO}_7(F)\backslash \mathrm{SO}_8(\mathbb{A})} \varphi_\pi(h)$$

$$\int_{F^{10}\backslash \mathbb{A}^{10}} f(\begin{pmatrix} 1 & u & * \\ & I_{10} & u' \\ & & 1 \end{pmatrix} \epsilon \begin{pmatrix} I_2 & & \\ & h & \\ & & I_2 \end{pmatrix})\psi^{-1}(u_1)dudh,$$

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where ϵ has the form $diag(1, \epsilon', 1)$; it is a certain rational lower unipotent element, which commutes with SO_7.

We use the fact that, for $f \in \Theta_\tau$, the following function on $\mathrm{SO}_{10}(\mathbb{A})$ is left invariant to the Adele points of the unipotent radical of the standard parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_1 \times \mathrm{SO}_8$

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$$g \mapsto \int_{F^{10}\backslash\mathbb{A}^{10}} f(\begin{pmatrix} 1 & u & * \\ & I_{10} & u' \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix})\psi^{-1}(u_1)du.$$

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$$\int_{V(F)\backslash V(\mathbb{A})} f(\begin{pmatrix} 1 & x & u & * & * \\ & 1 & v & * & * \\ & & I_8 & v' & u' \\ & & & 1 & -x \\ & & & & 1 \end{pmatrix} \epsilon \begin{pmatrix} I_2 & & \\ & h & \\ & & I_2 \end{pmatrix}) \psi^{-1}(u_1) du dh,$$

This can be written as

$$\int_{\mathrm{SO}_7(F)\backslash \mathrm{SO}_8(\mathbb{A})} \varphi_\pi(h)(f^{U_2})^\psi(\epsilon \begin{pmatrix} I_2 & & \\ & h & \\ & & I_2 \end{pmatrix})dh,$$

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Factor integration through $\mathrm{SO}_7(\mathbb{A})$.

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Factor integration through $\mathrm{SO}_7(\mathbb{A})$.

$$\int_{\mathrm{SO}_7(\mathbb{A})\backslash \mathrm{SO}_8(\mathbb{A})} \int_{\mathrm{SO}_7(F)\backslash \mathrm{SO}_7(\mathbb{A})} \varphi_\pi(gh)(f^{U_2})^\psi(g\epsilon \begin{pmatrix} I_2 & & \\ & h & \\ & & I_2 \end{pmatrix})dgdh.$$

We show that this is identically zero iff the inner integral is identically zero.

We prove that the space of functions on $\mathrm{SO}_7(\mathbb{A})$ generated by the restrictions to $\mathrm{SO}_7(\mathbb{A})$ of the functions $(f^{U_2})^\psi$, as f varies in Θ_τ, is equal to

the residual Eisenstein series $\Theta"_\tau$ corresponding to the parabolic induction from $\tau|\det\cdot|^s \otimes \sigma_\tau$, at $s = 1$;

σ_τ is the descent of τ to $\mathrm{SO}_3(\mathbb{A})$.

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Since $L^S(\pi \times \tau, s)$ has a pole at $s = \frac{3}{2}$, we conclude that the last integral is nontrivial. This proves that the ψ-Whittaker coefficient of σ is nontrivial and hence σ is nontrivial.